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Conjecture for Lower Bounds of Achromatic Indices of Specially Constructed Trees.

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ABSTRACT

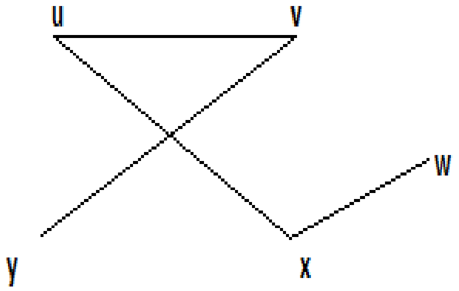
The bounds of an achromatic index of simple graph G are not known in general .In this paper we have conjectured lower bounds of achromatic indices of specially constructed trees.

KEY WORDS

Achromatic Index, Trees, Colouring of graphs

INTRODUCTION:

- A graph G consist of a finite nonempty set V(G) of p points together with a prescribed set E(G) of q unordered pairs of distinct points of V. Each pair e= [U, V] or UV of points in E(G) is a line of G or edge of G and e is said to join U and V. We say in such case that U and V are adjacent points. Point U and line e are incident with each other as are V and e. A graph G is called G(p q) graph. p is called order of a graph and q is called size of graph. It is customary to represent a graph by means of diagram and to refer to it as the graph.



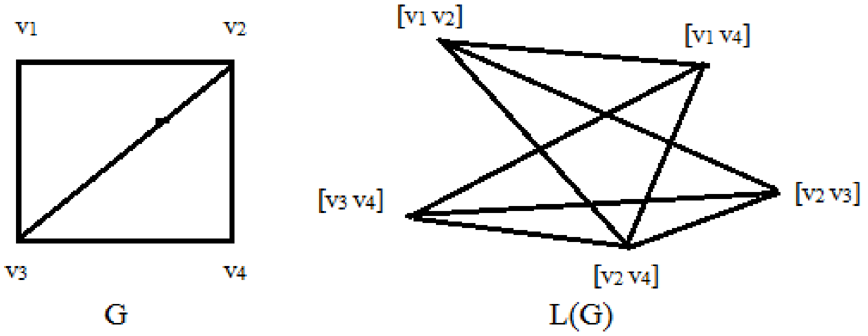
Thus we have U and V are adjacent but U and W are not adjacent to each other. The lines e_1=UX and e_2=VY intersects each other but their intersection is not a point of a graph. The lines UX and VY are adjacent to each other but UV and WX are not adjacent to each other.

- If a vertex is adjacent to itself (via some edge) then the edge is called a loop. If two vertices are joined by two or more different lines then the edges are called multiple edges. A graph without loops, multiple edges is called a simple graph[3].

- Degrees: The degree of a point V_i in graph G denoted by d_i or deg V_i is the number of lines incident at the vertex V_i.

- Walk and connectedness: A walk of a graph G is an alternating sequence of points and lines beginning and ending with points in which each line is incident with the two points immediately preceding and following it. Sometimes it can also be denoted only by points where the lines are evident by context. It is closed if its beginning point and last point are same otherwise it is open. It is a trail if all lines are distinct and it is a path if all points are distinct. A closed trail is called a circuit and a closed path is called a cycle.A graph is connected if every pair of points is joined by a path.

- Tree: A connected graph G without a cycle. For a tree T(p, q) it is known that p=q+1
- Matching: A matching of size k in a graph G is a set of k- pair-wise disjoint edges.
- The line graph of a graph G, written L(G), is the graph whose vertices are edges of G with ef ∈ E(L(G)) whenever e=UV, f=VW in E(G) for some U, V, W in V(G)[4]. For example



- Colouring of graphs: A k-colouring (vertex) of G is a labeling f :V(G) →{1,2…. k}. The labels are colour class. A k-colouring of f is proper if UV ∈ E(G) ⇒ f(U)≠f(V). A graph G is k-colourable if it has a proper k-colouring. A complete vertex colouring: A k-colouring of graph G is complete if for every pair of colours i, j (j≠j) there are adjacent points u and v coloured with these colours. The number ψ(G) is called achromatic number of graph G if G has ψ(G) complete (vertex) colouring and G can't have ψ (G)+1 complete colouring.

- The chromatic index: A k-edge colouring of G is a labeling f: E(G)→{1,2,… k} the labels are colours and the set of edges with one colour is a colour class. A k-edge colouring is proper if edges sharing a vertex have different colours, equivalently, each colour class is matching. A graph is k-edge colourable if it has a proper k-edge colouring. The edge chromatic number of a loop less graph G is the least k such that G is k-edge colourable. The chromatic index is denoted by χ′(G). If for a given proper edge colouring C if for any two pair of colours i, j there exist edges with colours i and j and sharing a common vertex then the colouring C is called a complete colouring of

G. The achromatic index $\psi'(G)$ of graph G is the number such that G has $\psi'(G)$ complete edge colouring and G can't have $\psi'(G)+1$ complete edge colouring[5].

1. It is known that for a simple graph G(p, q) whose degree sequence is $d_1\ d_2\ d_3 \ldots . d_p$ then
   $\psi'(G)\,(\psi'(G)-1) \square \sum {}^{d_i}P_2$ where the sum is taken over i=1to p [1]

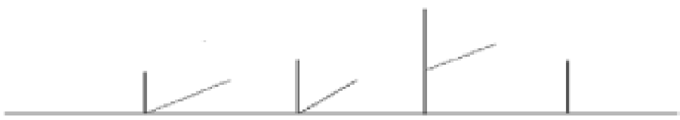
2. D. Geller, H. Kronk proved [2] that If m= max { n: $\square\frac{n-1}{2}\square.\ n \leq k$ } then,

   For $\psi(P_k)$ = m-1 if m odd and k=$\square\frac{m-1}{2}\square$. m
   = m otherwise
   and $\psi(C_k)$ = m-1 if m odd and k=$\square\frac{m-1}{2}\square$. m +1
   = m otherwise

It is clear that $\psi(P_k) = \psi'(P_{k+1})$
It is natural to have for any tree T(p, q), $\psi'(P_{\text{length longest path of }T}) \leq \psi'(T) \leq q$

THE SPECIAL CONSRUCTION OF BUSHES:

Above we have seen algorithm to construct complete colouring for the given graph, but algorithms are also useful to construct certain graphs. We will define a tree to be bushes if its longest path is horizontally drawn and all the branches can be drawn upwards and for any branch, if V is upper vertex and U is the lower vertex on the path and U, V are not pendant vertices then deg V $\leq$ deg U and degrees of every vertical path and longest path are evenly distributed except pendant vertices. For example



If you want to draw a tree T(p, q) having λ to be length of longest path then it can be drawn as a bushes and the procedure is describe below.
Without loss of generality Let length of longest path in T = λ.
At any step while drawing the edges, draw available edges in the mentioned order whenever possible and the following process terminates if q edges are drawn.
Case i) If $\lambda = 2m$ (i.e. λ is even)
Step 1: Label the vertices of the longest path to be $V_1, V_2, V_3, \ldots, V_m, V_{m+1}, \ldots, V_{2m+1}$ and draw this path horizontally on the paper.

Draw a vertical path (we will call it as stem) of length $\frac{\lambda}{2}$ standing on the vertex V_{m+1}. Label the vertices from bottom to top as V_{m+1} (which is already there), MV_1, MV_2, …………., MV_m

Now draw a stem of length $\frac{\lambda}{2}$ -1 standing on the vertex V_{m+2}.Label the vertices from bottom to top as V_{m+2} (which is already there), R_2V_1, R_2V_2, …..,R_2V_{m-1}

Now draw a stem of length $\frac{\lambda}{2}$ -1 standing on the vertex V_m. Label the vertices from bottom to top as V_m (which is already there), L_2V_1, L_2V_2, …….L_2V_{m-1}

Now draw a stem of length $\frac{\lambda}{2}$ -2 standing on the vertex V_{m+3}. .Label the vertices from bottom to top as V_{m+3} (which is already there), R_3V_1, R_3V_2, ……,R_3V_{m-2}

Now draw a stem of length $\frac{\lambda}{2}$ -2 standing on the vertex V_{m-1}. .Label the vertices from bottom to top as V_{m-1} (which is already there), L_3V_1, L_3V_2, …..,L_3V_{m-1}

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Now draw a stem of length 1 standing on the vertex V_{2m}.Label the standing pendant vertex as R_mV_1

Now draw a stem of length 1 standing on the vertex V_2. Label the standing pendant vertex as L_mV_1

Step 2:

Draw a leaf at each of the vertex MV_1, R_2V_1, L_2V_1, ………, $R_{m-1}V_1$, $L_{m-1}V_1$

Draw a leaf at each of the vertex MV_2, R_2V_2, L_2V_2, ………., $R_{m-2}V_2$, $L_{m-2}V_2$

Draw a leaf at each of the vertex MV_3, R_2V_3, L_2V, ………$R_{m-3}V$, $L_{m-3}V_3$

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Draw a leaf at the vertex MV_{m-1}

Step 3: Draw a leaf at each of the vertex V_{m+1}, V_{m+2}, V_m, V_{m+3}, V_{m-1},…..,V_{2m}, V_2

Step 4: Repeat Step 2 and Step 3 until all q edges are drawn.

Case ii) If $\lambda = 2m-1$ (i.e. λ is odd)

Step 1:Label the vertices of the longest path to be V_1, V_2, V_3, …,V_m,V_{m+1},…,V_{2m} and draw this path horizontally on the paper.

Draw a vertical path of length $\frac{\lambda-1}{2}$ standing on the vertex V_{m+1}. Label the vertices from bottom to top as V_{m+1} (which is already there), R_1V_1, R_1V_2, …………..,R_1V_{m-1}

Draw a vertical path of length $\frac{\lambda-1}{2}$ standing on the vertex V_m. Label the vertices from bottom to top as V_m (which is already there),L_1V_1, L_1V_2, …………..,L_1V_{m-1}

Now draw a stem of length $\frac{\lambda-1}{2}$ -1 standing on the vertex V_{m+2}. Label the vertices from bottom to top as V_{m+2} (which is already there), R_2V_1, R_2V_2, …..,R_2V_{m-2}

Now draw a stem of length $\frac{\lambda-1}{2}$ -1 standing on the vertex V_{m-1}.Label the vertices from bottom to top as V_{m-1} (which is already there), L_2V_1, L_2V_2, …..,L_2V_{m-2}

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Now draw a stem of length 1 standing on the vertex V_{2m-1}. Label the standing pendant vertex as $R_{m-1}V_1$

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Now draw a stem of length 1 standing on the vertex V_2. Label the standing pendant vertex as $L_{m-1}V_1$

Step 2:

Draw a leaf at each of the vertex R_1V_1, L_1V_1, ………,$R_{m-2}V_1$, $L_{m-2}V_1$

Draw a leaf at each of the vertex R_1V_2, L_1V_2, ………,$R_{m-3}V_2$, $L_{m-3}V_2$

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Draw a leaf at the vertex R_1V_{m-2}, L_1V_{m-2}

Step 3: Draw a leaf at each of the vertex V_{m+1}, V_m, V_{m+2}, V_{m-1}, V_{m+3},V_{m-2}…..V_{2m-1}, V_2

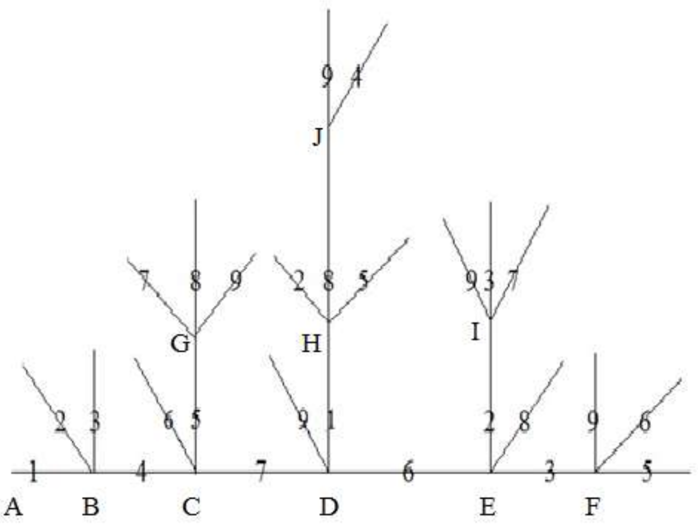
Step 4: Repeat Step 2 and Step 3 until all q edges are drawn.

The graph drawn in this constructive way is Bushes graph. The spirit taken from the result of achromatic index of P_n as mentioned in the chapter 2. Surprisingly gives the Lower bound for the bushes conjectured as follows.

THE CONJECTURE

Let d_1,d_2 , d_3,…………………,d_{q+1} be the degree sequence of a bushes graph. If for natural number x, $x.(x-1) \leq \sum {}^{d_i}P_2$ and $\sum {}^{d_i}P_2 \leq x.(x+1)$ then a lower bound for the achromatic index of the bushes graph is x-1.

The proof of the result is not established but the result sounds like result 2 discussed in the introduction & the result found to be true by manual colouring for many bushes, one such example is given below. The numbers on the edges are colours.



For the bush above we have λ=6 q=27

The degree sequence of the graph is {4, 4, 4, 4, 4, 4, 4, 4, 3, 1,1………19 times}

For each degree d_i , Summing the factors ${}^{d_i}P_2$ we get the sum to be 102

The value of x according to the conjecture is 10

Hence a lower bound of achromatic index of the graph is 9.

The colouring is complete due to following table.

Colour Pair	Adjacent vertex	Colour Pair	Adjacent vertex
(1 2)	A	(3 7)	I
(1 3)	A	(3 8)	E
(1 4)	A	(3 9)	I
(1 5)	H	(4 5)	C
(1 6)	D	(4 6)	C
(1 7)	D	(4 7)	C
(1 8)	H	(4 8)	J
(1 9)	D	(4 9)	J
(2 3)	A	(5 6)	C
(2 4)	A	(5 7)	C
(2 5)	H	(5 8)	H
(2 6)	E	(5 9)	F
(2 7)	I	(6 7)	C
(2 8)	H	(6 8)	E
(2 9)	I	(6 9)	F
(3 4)	A	(7 8)	G
(3 5)	F	(7 9)	I
(3 6)	E	(8 9)	G

From the natural upper bound[1] result, it is clear that the upper bound for the achromatic index of bushes graph is x mentioned in the above conjecture.

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