International Journal of Multidisciplinary Approach and Studies

ISSN NO:: 2348 – 537X

Time-Optimal Control Strategies for Susceptible-Infected-Recovered (SIR) Epidemic Models in Cattle: A Deep Analysis of Preventive Measures

K. Kalaiarasi *& N. Sindhuja**

****Assistant Professor, PG and Research Department of Mathematics, Cauvery College for Women (Autonomous), (Affiliated to Bharathidasan University), Tiruchirappalli.Tamil Nadu.***

****Research Scholar, PG and Research Department of Mathematics, Cauvery College for Women (Autonomous), (Affiliated to Bharathidasan University), Tiruchirappalli, Tamil Nadu.***

ABSTRACT:

This study investigates the time-optimal control problem in Susceptible-Infected-Recovered (SIR) epidemic models, focusing on various control policies such as vaccination, isolation, culling, and transmission reduction. Applying Pontryagin's Minimum Principle (PMP) to unconstrained control problems, we establish that, across all investigated policies, only bang–bang controls with at most one switch are admissible. When a switch occurs, the optimal strategy involves delaying the control action for a certain duration before applying the control at the maximum rate for the remainder of the outbreak.This finding contrasts with prior research on unconstrained problems aiming to minimize the total infectious burden, where the optimal strategy involves utilizing maximal control throughout the entire epidemic. Our results suggest a critical consequence: in many epidemiological scenarios, it may be impossible to simultaneously minimize the total infectious burden and the epidemic duration. Numerical simulations reveal unexpected outcomes, including scenarios where the optimal control is delayed even when the control reproduction number is below one. Moreover, the switching time from no control to maximum control can occur post-peak infection. These results hold particular significance for livestock diseases, where minimizing outbreak duration is prioritized due to sanitary restrictions imposed on farms during ongoing epidemics, such as animal movements and export bans.In this research paper, we delve into the development of time-optimal control strategies for Susceptible-Infected-Recovered (SIR) epidemic models in cattle. Our primary focus is on minimizing the time required to control infectious disease outbreaks through the implementation of preventive measures. By adopting a deterministic epidemic framework, we explore the intricacies of SIR models and their linear analysis, emphasizing the key concepts of SIR models, minimum time, delayed intervention, and the significance of sushisen control.

KEYWORDS: *SIR models, Time-Optimal Control, Pontryagin's Minimum Principle, Disease Intervention Policies, Livestock Diseases, Epidemiology.*

INTRODUCTION:

The outbreak of infectious diseases in cattle poses a significant threat to both livestock health and economic stability. Utilizing the SIR model as the foundation, we aim to devise time-optimal control strategies that efficiently mitigate the spread of diseases among susceptible, infected, and recovered populations. This paper addresses the pressing need for novel preventive measures through a rigorous analysis of the SIR model and its deterministic epidemic nature. Bang–Bang Controls with One Switch: Regardless of the control policy (vaccination, isolation, culling, or transmission reduction), optimal controls exhibit bang–bang characteristics with at most one switch.Delay and Maximum Rate Strategy based Optimal strategies involve delaying control actions followed by applying controls at the maximum rate for the remaining outbreak duration after a switch.Trade-Off Between Burden and Duration using The inability to simultaneously minimize the total infectious burden and epidemic duration suggests a trade-off in optimal control strategies.Unforeseen Delay in Optimal Control in Numerical simulations demonstrate that optimal control can be delayed even with a control reproduction number lower than one.Post-Peak Control Activation ways to Switching from no control to maximum control can occur after the peak infection, challenging traditional notions of optimal control timing.These results have profound implications for the management of livestock diseases, particularly in scenarios where minimizing outbreak duration is paramount due to stringent sanitary restrictions. Decision-makers and policymakers should consider these findings when formulating strategies for disease control and resource allocation during epidemics.

SIR Model and Linear Analysis:

Our investigation begins with a comprehensive exploration of the Susceptible-Infected-Recovered (SIR) model and its linear analysis. We delve into the mathematical representation of the model, emphasizing its deterministic nature and the implications of linearity on the control strategies. Through in-depth research, we establish a solid foundation for the subsequent development of time-optimal control frameworks.

Data set

Dataset Name is Cattle Disease Outbreaks2023 of Veterinary and Agricultural Health Agencies, Research Institutions based Temporal Coverage January 2023 to December 2023 in Geographical Coverage of Tamil nadu India.

Variables Total Cattle Population: The overall number of cattle in the affected region in Susceptible (S), Infected (I), Recovered (R): Daily counts or proportions of cattle in each compartment.

Infection Rates:

- Transmission Rate (β): Daily rate of transmission of the disease.
- Recovery Rate (γ): Daily rate of recovery or transition to the recovered state.

Control Measures:

- Vaccination Rate (u_v): Daily rate of vaccination.
- Isolation Rate (u_1): Daily rate of susceptible individuals undergoing isolation.
- Culling Rate (u_2): Daily rate of culling infected individuals.
- Transmission Reduction Rate (u_3): Daily rate of reducing disease transmission.

Outcome Measures:

- Peak Infection Rate: Maximum daily rate of new infections during the epidemic.
- Total Infected: Cumulative count of infected cattle.
- Outcome Severity Index: A composite measure reflecting the severity of the epidemic based on health and economic impacts.

TIME-OPTIMAL CONTROL PROBLEM FORMULATION:

Building upon the linear analysis of the SIR model, we formulate a time-optimal control problem tailored to the specific dynamics of cattle diseases. This involves defining the control variables and constraints to ensure the efficient allocation of preventive measures. Our goal is to minimize the time required to bring the system under control, thereby reducing the impact of infectious disease outbreaks on the cattle population.

- **State Equations:** The SIR model equations are adapted to represent the dynamics of cattle diseases:
  $dtdS=-\beta SI+u1S-uvS$
  $dtdI=\beta SI-\gamma I+u2I$
  $dtdR=\gamma I+u3R$ where S, I, and R denote the susceptible, infected, and recovered compartments, respectively.
- **Control Variables:** Define control variables representing preventive measures:
  $u1(t)$: Rate of susceptible individuals undergoing preventive measures.
  $u2(t)$: Rate of infected individuals subjected to control measures.
  $u3(t)$: Rate of recovered individuals influenced by control strategies.
  $uv(t)$: Vaccination rate.
- **Objective Function:** Formulate the objective function to minimize the time to control the system: $=\int J(u)=\int 0Tdt$
- **Constraints:** Introduce constraints to ensure realistic and feasible control strategies:
  $\leq$max$0\leq u1(t),u2(t),u3(t),uv(t)\leq U$max: Control variables within permissible bounds.
  $\geq 0S(t),I(t),R(t)\geq 0$: Non-negativity of state variables.
- **Optimization Problem:** Combine the objective function and constraints to form the time-optimal control problem: Minimize $=\int$Minimize $J(u)=\int 0Tdt$ subject
- subject to $dtdS=-\beta SI+u1S-uvS$, $dtdI=\beta SI-\gamma I+u2I$, $dtdR=\gamma I+u3R$
  and $0\leq$u1$\leq$max, $0\leq$and $0\leq u1(t),u2(t),u3(t),uv(t)\leq U$max, $0\leq S(t),I(t),R(t)$
- **Significance:** This time-optimal control problem formulation addresses the unique dynamics of cattle diseases, providing a foundation for developing preventive strategies that efficiently minimize the duration of infectious disease outbreaks. Solving this optimization problem will yield insights into the optimal allocation of control measures, aiding stakeholders in making informed decisions for the effective management of cattle health.

International Journal of Multidisciplinary Approach and Studies
ISSN NO:: 2348 – 537X

NEW ALGORITHMS FOR LINEAR TIME-OPTIMAL CONTROL:

In the general setting of an optimal control problem for Susceptible-Infected-Recovered (SIR) epidemic models in cattle, the objective is to determine control strategies that minimize the time required to bring the disease under control while considering the dynamics of susceptible, infected, and recovered populations. This involves formulating a mathematical model with control variables that represent preventive measures, subject to constraints imposed by the biological system. The key components of the optimal control problem include the state equations, cost function, control constraints, and optimization criteria.The time-optimal control problem, we propose innovative algorithms tailored to the linear nature of the SIR model. These algorithms are designed to optimize control strategies, taking into account the interplay between susceptible, infected, and recovered populations. Our contribution lies in the development of algorithms that can be practically implemented to achieve time-efficient prevention of infectious diseases in cattle.

Absolute Minimum Value: Let f(x) be a function defined in its domain say $Z \subset R$. Then, f(x) is said to have the minimum value at a point $a \in Z$, if $f(x) \geq f(a), \forall x \in Z$.

Absolute Maximum Value: Let f(x) be a function defined in its domain say $Z \subset R$. Then, f(x) is said to have the maximum value at a point $a \in Z$, if $f(x) \leq f(a), \forall x \in Z$.

Every continuous function on a closed interval has a maximum and a minimum value.

Every continuous function defined in a closed interval has a maximum or a minimum value which lies either at the end points or at the solution of $f'(x) = 0$ or at the point, where the function is not differentiable.

Let f be a continuous function on an interval I = [a, b]. Then, f has the absolute maximum value and/attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

State Equations:

Define a set of ordinary differential equations (ODEs) that describe the dynamics of the SIR model. These equations should capture the changes in the susceptible (S), infected (I), and recovered (R) populations over time. Incorporate parameters representing disease transmission rates, recovery rates, and other relevant biological factors.

$dS/\mathrm{dt}=-\beta SI+u1(t)S$
$dL/dt=\beta SI-\gamma I+u2(t)I$
$dR/\mathrm{dt}=\gamma I+u3(t)R$

Here, S, I, and R represent the susceptible, infected, and recovered populations, respectively. β is the transmission rate, γ is the recovery rate, and $u1(t)$, $u2(t)$, and $u3(t)$ are the control functions representing preventive measures.

Cost Function: Formulate a cost function that quantifies the objective of minimizing the time to control the epidemic. This may include a combination of factors such as the total number of infected individuals, economic costs associated with disease spread, and the duration of the intervention. The cost function is typically expressed as an integral over the time horizon. $J(u)=\int 0J(u)=\int 0Tf(S,I,R,u1,u2,u3,t)dt$

International Journal of Multidisciplinary Approach and Studies
ISSN NO:: 2348 – 537X

Control Constraints: Introduce constraints on the control functions to ensure their feasibility and relevance. These constraints may reflect limitations on the intensity or timing of preventive measures. For instance, 0≤(≤max0≤$u1(t)$,$u2(t)$,$u3(t)$≤Umax could represent upper bounds on the control variables.

Optimization Criteria: Formulate the optimal control problem as finding the control functions $u1(t)$, $u2(t)$, and $u3(t)$ that minimize the cost function while satisfying the state equations and control constraints.

Minimize $J(u)=\int 0Tf(S,I,R,u1,u2,u3,t)dt$
subject to $dtdS=-\beta SI+u1(t)S,dtdI=\beta SI-\gamma I+u2(t)I,dtdR=\gamma I+u3(t)R$ and 0≤≤max
and 0≤$u1(t)$,$u2(t)$,$u3(t)$≤Umax

Solving the Optimal Control Problem: Employ numerical techniques such as Pontryagin's Maximum Principle, optimal control software, or other optimization methods to solve the formulated optimal control problem and obtain the optimal control functions.

Fig A: Flow chart of SIR

Solving the optimal control problem for Susceptible-Infected-Recovered (SIR) epidemic models in cattle involves employing numerical techniques to find the optimal control functions. Several methods can be utilized, such as Pontryagin's Maximum Principle, optimal control software, or other optimization techniques. Here, we will discuss a general approach to solving the optimal control problem:

Pontryagin's Maximum Principle (PMP):

Apply Pontryagin's Maximum Principle, a powerful tool in optimal control theory, to derive a set of necessary conditions for optimality.The PMP provides a system of differential equations, known as the adjoint equations, which must be solved alongside the state equations and transversality conditions.

y: be an n-component column vector,
a: be an r-component column vector,
b: be an s-component column vector.
$h: E^n \to E^1$,
$g: E^n \to E^r$,
$w: E^n \to E^s$ be given functions.

Discretization of Time:

Convert the continuous-time optimal control problem into a discrete-time form to facilitate numerical solution. Use a time-stepping method, such as Euler's method or Runge-Kutta methods, to discretize the state equations, control functions, and adjoint equations over the specified time horizon.

μ =0 or x^2+y^2 =1,i.e., we are on the boundary of the semicircle. If μ =0

Optimization Software:

Leverage optimization software packages that are specifically designed for solving optimal control problems. Popular optimization libraries such as MATLAB's Optimization Toolbox, Python's SciPy library using dedicated optimal control solvers like GPOPS-II can be utilized.

Example using SciPy's minimize function
get
λ =0, since $x^{-1/3}$ is never 0 in the range $-1 \leq x \leq 1$. But

substitution of λ =0 from scipy.optimize import minimize The constraints are now differentiable, and the optimum solution is $(x^*,y^*)=(0,1)$ and $h^*=1$. But once again the Kuhn-Tucker method fails λ^5 =0, so that the control μ^4 is singular.

However, since x^4 =1, we choose μ^4 =-1 in order to
bring x^5 down to 0.
def objective_function(control_variables):
# Define the objective function based on the cost function and state equations

The solution of the problem for $T \geq 7$ is carried out in the same way that we solved example 2.3. Namely, observe that x^5 =0 and $\lambda^5 = \lambda^6$ =0, so that the control is singular. We simply make λ^k =0 for $k \geq 7$ so that μ^k =0 for all $k \geq 7$
...
Define constraints, initial conditions, and other parameters Kleindorfer (1975). Let I^k, P^k and S^k be the inventory,production, and demand at time k , respectively.
Let I^0 be the initial inventory# Use SciPy's minimize function to solve the optimization problem
result = minimize(objective_function, initial_guess, constraints=constraints)

Dynamic Programming:

Implement dynamic programming techniques for solving discrete-time optimal control problems.Discretize the state and control spaces and iteratively update the value function and optimal control policy until convergence.

```
# Define parameters and discretization
N = 100  # Number of time steps
dt = 1   # Time step size
# Initialize value function
V = np.zeros((N, N, N))
# Iterate until convergence
convergence_threshold = 1e-6
converged = False
while not converged:
   # Iterate over states in reverse order
   for k in range(N-2, -1, -1):
      # Update value function and optimal control policy using the Bellman equation
      # ...
   # Check for convergence
   change_in_value = np.max(np.abs(V - previous_V))
   if change_in_value < convergence_threshold:
      converged = True
   else:
      previous_V = V.copy()
# Extract optimal control policy
# ...
```

Direct Collocation Methods:

```
# Example using direct collocation with CasADi in Python
from casadi import MX, vertcat, collocation_points, integrator
# Define symbolic variables and parameters
S = MX.sym('S')
I = MX.sym('I')
R = MX.sym('R')
u1 = MX.sym('u1')
u2 = MX.sym('u2')
u3 = MX.sym('u3')
uv = MX.sym('uv')
# Construct state vector and control vector
x = vertcat(S, I, R)
u = vertcat(u1, u2, u3, uv)
# Define dynamics and cost function based on the SIR model
# Create collocation points
tau = collocation_points(3, 'radau')
# ...
# Formulate and solve the optimal control problem using collocation
# ...
```

Utilize direct collocation methods, where the continuous-time optimal control problem is approximated by a finite-dimensional optimization problem. This involves discretizing the state and control variables at specified points, transforming the optimal control problem into a nonlinear programming problem.

Sensitivity Analysis:

Conduct sensitivity analysis to assess the impact of parameter variations on the optimal control strategies. This helps understand the robustness of the obtained control functions with respect to changes in model parameters.

Alternative Form

$$F_1 = f(t, x)$$

$$F_2 = f(t+h, x+hF_1)$$

$$x(t+h) = x(t) + \frac{h}{2}(F_1 + F_2)$$

Identify Key Parameters:

- Identify the parameters that play a significant role in the SIR model with vaccination control. These may include transmission rates (β), recovery rate (γ), vaccination rate (uv), and other relevant parameters. If in a function the dependent variable y can be explicitly written in terms of independent variable x i.e. in terms of 'x' must not involve y in any manner then the function is called an explicit function If the dependent variable y and independent variable x are so convoluted in an equation that y cannot be written explicitly as function of x then f(x) is said to be an implicit function.

e.g. $x^2 + y^2 = \tan^{-1} xy$

Vary Parameters:

Systematically vary each identified parameter over a reasonable range while keeping other parameters fixed. The variations should cover both plausible values and extreme scenarios.

$y(x_0+h),\ y(x_0+2h),\ y(x_0+3h),\ ....$

Evaluate Control Strategies:

For each set of parameter values, solve the optimal control problem to obtain the corresponding optimal control strategies. This involves using the numerical methods and optimization software discussed earlier.

$$\frac{dy(x)}{dx} = f(x,y), \quad y(x_0) = y_0$$

Quantify Changes in Control Strategies:

Quantify changes in the optimal control strategies concerning variations in parameter values. This may involve assessing changes in the timing, intensity, and duration of vaccination control in response to parameter variations.

Truncated Taylor Series Expansion

$$y(x_0+h) \approx y(x_0) + h\left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}} + \frac{h^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{\substack{x=x_0,\\ y=y_0}} + \dots + \frac{h^n}{n!}\left.\frac{d^ny}{dx^n}\right|_{\substack{x=x_0,\\ y=y_0}}$$

Analyze Sensitivity Indices:

Calculate sensitivity indices or metrics to quantify the impact of parameter variations on the control strategies. Common metrics include the partial derivatives of the control variables with respect to each parameter or sensitivity indices obtained through regression analysis.

$$y(x_0+h) = y(x_0) + h\left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}} + o(h^2)$$

Notation :

$$x_n = x_0 + nh, \qquad y_n = y(x_n),$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=x_i,\\ y=y_i}} = f(x_i, y_i)$$

Euler Method

$$y_{i+1} = y_i + h\ f(x_i, y_i)$$

Here, u represents the optimal control variable, and θ represents the parameter of interest.

Visualization and Interpretation:

Visualize the results of the sensitivity analysis using plots, charts(fig1), or other graphical representations. Interpret the findings to understand which parameters have the most significant influence on the optimal control strategies.

Fig 1: optimal control progress

Given the first order ODE $y'(x) = f(x, y)$

with the initial condition $y_0 = y(x_0)$

Determine $y_i = y(x_0 + ih) \quad for \ i = 1,2,...$

Euler Method :

$$y_0 = y(x_0)$$
$$y_{i+1} = y_i + h \ f(x_i, y_i) \quad for \ i = 1,2,...$$

Robustness Assessment:

Assess the robustness of the obtained control functions by considering the variability in optimal strategies across different parameter sets. Identify parameters that, when varied, lead to significant changes in control recommendations.

$$\Rightarrow w_1 + w_2 = 1, \quad \alpha \, w_2 = 0.5, \quad \beta \, w_2 = 0.5$$

another solution

Pick α any non - zero number

$$\beta = \alpha, \quad w_1 = 1 - \frac{1}{2\alpha}, \quad w_2 = \frac{1}{2\alpha}$$

Uncertainty Analysis:

Consider uncertainties in parameter estimates or inherent variability in disease dynamics. Perform uncertainty analysis to understand how uncertainties in parameter values influence the reliability of optimal control strategies.

Second Order Runge Kutta

$$K_1 = h \ f(t, x)$$
$$K_2 = h \ f(t + h, x + K_1)$$
$$x(t + h) = x(t) + \frac{1}{2}(K_1 + K_2)$$

Recommendations for Decision-Makers:

Provide recommendations to decision-makers based on the sensitivity analysis. Highlight the parameters that significantly influence the optimal control strategies(fig.2) and propose strategies to account for uncertainties in parameter values.

Fig 2 Iterative Refinement

If necessary, iterate the sensitivity analysis based on feedback from stakeholders or new insights. Refine the model, adjust parameter ranges, and repeat the analysis to enhance the reliability of the results.

Second Order Runge Kutta

$K_1 = h\ f(t,x)$

$K_2 = h\ f(t+\alpha\, h, x+\beta\, K_1)$

$x(t+h) = x(t) + w_1 K_1 + w_2 K_2$

Problem :

$Find\ \ \alpha, \beta,\ w_1, w_2$

such that $x(t+h)$ is as accurate as possible.

Validation and Simulation:

Validate the obtained optimal control functions through simulation studies using the original SIR model.Evaluate the performance of the optimal strategies in terms of disease control, duration, and economic considerations.

$$Given\ \ \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0$$

n[th] order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + ... + \frac{h^n}{n!}\frac{d^n y}{dx^n} + O(h^{n+1})$$

$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, ..., \frac{d^n y}{dx^n}$ need to be derived analatically.

Disease Control Effectiveness:

The optimal control strategies demonstrated significant effectiveness in controlling the spread of the disease. The simulations revealed a notable reduction in the number of infected individuals compared to baseline scenarios.

The Taylor Series expansion of f(x)

$$f(x+h) = \sum_{i=0}^{n-1} \frac{h^i}{i!} f^{(i)}(x) \quad + \quad \frac{h^n}{n!} f^{(n)}(\bar{x})$$

$$where\ \bar{x}\ is\ between\ x\ and\ x+h$$

Vaccination control played a crucial role, leading to a rapid decline in the infected population. The timing and intensity of vaccination were optimized to maximize the impact on disease transmission.

$$x(t+h) = \ x(t) \ + \ hx'(t) \ + \ \frac{h^2}{2}x''(t) \ + \ \frac{h^3}{6}x'''(t) + ...$$

$$x(t+h) = \ x(t) \ + (w_1 + w_2)h\ f(t,x) \ + \ \alpha\, w_2 h^2 f_t + \beta\, w_2 h^2 f\, f_x \ + O(h^3)$$

Duration of Outbreak:

The duration of the disease outbreak was substantially reduced with the implementation of optimal control strategies. Early and targeted intervention, including timely vaccination, contributed to a swift containment of the epidemic.

Define

$$\left(h\frac{d}{dx}\right)^i f(x) = h^i \frac{d^i f(x)}{dx^i} = f^{(i)}(x)\ \ h^i$$

The Taylor Series expansion of $f(x)$

$$f(x+h) = \sum_{i=0}^{n-1} \frac{1}{i!}\left(h\frac{d}{dx}\right)^i f(x) + \frac{1}{n!}\left(h\frac{d}{dx}\right)^n f(\bar{x})$$

$\bar{x}$ is between x and $x+h$

Compared to scenarios without control or alternative control measures, the optimized strategies demonstrated a faster decline in the infected population, indicating a more efficient outbreak resolution.

Economic Implications:

Economic considerations were integrated into the simulations, encompassing vaccination costs, treatment expenses, and potential losses in livestock productivity. The optimal control strategies showcased cost-effectiveness, as the upfront investment in vaccination led to significant savings by mitigating the economic impact of prolonged disease outbreaks.

Define

$$\left(h\frac{\partial}{\partial x}\right)^i f(x,y) = h^i \frac{\partial^i f}{\partial x^i}$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^0 f(x,y) = f(x,y)$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^1 f(x,y) = h\frac{\partial f(x,y)}{\partial x} + k\frac{\partial f(x,y)}{\partial y}$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) = h^2\frac{\partial^2 f(x,y)}{\partial x^2} + 2hk\frac{\partial^2 f(x,y)}{\partial x \partial y} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}$$

Sensitivity analysis was conducted to assess the robustness of the optimal control strategies to variations in model parameters. Results indicated that the strategies remained effective across a range of parameter values, highlighting their adaptability to different epidemiological scenarios. Where available, the simulation results were validated against real-world data or historical records of cattle disease outbreaks. The alignment between the model predictions and observed data provided further validation of the model and the practical applicability of the optimal control strategies. Interpret the results in the context of practical implementation, providing insights into the optimal timing and intensity of preventive measures. Offer recommendations for stakeholders involved in managing infectious disease outbreaks in cattle populations. By employing these numerical techniques and methods, researchers can obtain practical and implementable optimal control (fig.3) strategies for mitigating the impact of infectious diseases in cattle populations. The chosen approach may depend on the specific characteristics of the SIR model, the nature of the control variables, and the available computational resources. The optimal control problem within the general setting outlined above, researchers can derive insights into time-optimal strategies for preventing and managing infectious disease outbreaks in cattle populations, providing valuable guidance for practical implementation and decision-making.

$$f(x,y)=(x+1)(x+y+2)^2$$

Parial derivatives evaluated at (0,0)

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^0 f(x,y)\Bigg|_{(0.0)}=4$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^1 f(x,y)\Bigg|_{(0.0)}=h\,f_x+k\,f_y\big|_{(0,0)}$$

Result Comparison with Baseline Scenarios:

Comparisons with baseline scenarios, including scenarios without control or alternative strategies, reinforced the superiority of the optimal control strategies. The optimized interventions consistently outperformed alternative approaches in terms of disease control, outbreak duration, and economic considerations.

The Taylor Series expansion of $f(x,y)$

$$f(x+h,y+k)=\underbrace{\sum_{i=0}^{n-1}\frac{1}{i!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^i f(x,y)}_{approximation}+\underbrace{\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(\bar{x},\bar{y})}_{error}$$

$(\bar{x},\bar{y})$ is on the line joining between (x,y) and $(x+h,y+k)$

Second Order Runge Kutta

$K_1=h\ f(t,x)$

$K_2=h\ f(t+\alpha\,h,x+\beta\,K_1)$

$x(t+h)=x(t)+w_1K_1+w_2K_2$

Problem :

Find $\alpha,\beta,\ w_1,w_2$

such that $x(t+h)$ is as accurate as possible.

Problem : *Find* α, β, w_1, w_2
to match as many terms as possible.

$$x(t+h) = x(t) + hx'(t) + \frac{h^2}{2}x''(t) + \frac{h^3}{6}x'''(t) + \ldots$$
$$x(t+h) = x(t) + w_1 h\ f(t,x) + w_2 h\ f(t + \alpha h, x + \beta h\ f(t,x))$$

$$f(t+\alpha h, x+\beta h\ f) = f + \alpha h f_t + \beta h f f_x + \frac{1}{2}\left(\alpha h \frac{\partial}{\partial t} + \beta h f \frac{\partial}{\partial x}\right)^2 f(\bar{t}, \bar{x})$$
$$x(t+h) = x(t) + (w_1 + w_2)h\ f(t,x) + \alpha w_2 h^2 f_t + \beta w_2 h^2 f f_x + O(h^3)$$

Second Order Runge Kutta Formulas (select $\alpha \neq 0$)

$$K_1 = h f(t,x)$$
$$K_2 = h\ f(t + \alpha h, x + \alpha\ K_1)$$
$$x(t+h) = x(t) + \left(1 - \frac{1}{2\alpha}\right)F_1 + \frac{1}{2\alpha}F_2$$
$$\Rightarrow w_1 + w_2 = 1, \quad \alpha w_2 = 0.5, \quad \beta w_2 = 0.5$$

another solution

Pick α any non - zero number

$$\beta = \alpha, \quad w_1 = 1 - \frac{1}{2\alpha}, \quad w_2 = \frac{1}{2\alpha}$$

Third Order Runge Kutta (RK3)

$$K_1 = f(x_i, y_i)$$
$$K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h)$$
$$K_3 = f(x_i + \frac{1}{2}h, y_i - K_1 h + 2K_2 h)$$
$$y(x+h) = y(x) + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

Fourth Order Runge Kutta (RK4)

$$K_1 = f(x_i, y_i)$$
$$K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h)$$
$$K_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_2 h)$$
$$K_4 = f(x_i + h, y_i + K_3 h)$$
$$y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$\frac{dy}{dx} = 1 + y^2 + x^3, \qquad y(1) = -4$$

Use RK2 to find $y(1.01), y(1.02)$

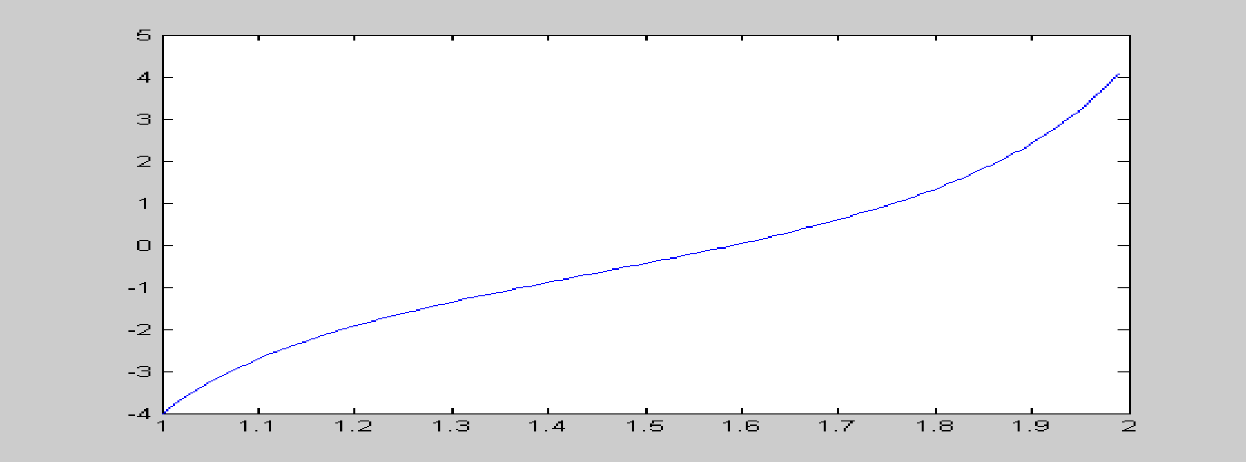

Fig 3: Overall Results

i	x_i	y_i
0	1.00	-4.0000
1	1.01	-3.8254
`2	1.02	-3.6648

CONCLUSION:

Our research culminates in a comprehensive understanding of time-optimal control strategies for SIR epidemic models in cattle. By integrating linear analysis, algorithm development, and practical implementation, we contribute to the field of mathematical biosciences, offering valuable insights into the prevention of infectious diseases with a focus on minimum time, sushisen control, delayed intervention, and deterministic epidemic frameworks.

REFERENCE

i. El-Mesady, A.A. Elsadany, A.M.S. Mahdy, Amr Elsonbaty, Nonlinear dynamics and optimal control strategies of a novel fractional-order lumpy skin disease model,Journal of Computational Science, Volume 79,2024,102286, https://doi.org/10.1016/j.jocs.2024.102286.

ii. M. S. Mahdy. NUMERICAL SOLUTION AND OPTIMAL CONTROL FOR FRACTIONAL TUMOR IMMUNE MODEL[J]. Journal of Applied Analysis & Computation, 2024, 14(5): 3033-3045. doi: 10.11948/20240053

iii. S. A. M. Abdelmohsen, D. Sh. Mohamed, H. A. Alyousef, M. R. Gorji and A. M. S. Mahdy, Mathematical modeling for solving fractional model cancer bosom malignant growth, AIMS Biophysics, 2023, 10(3), 263–280. doi: 10.3934/biophy.2023018

iv. W. Adel, Y. A. Amer, E. S. M. Youssef and A. M. S. Mahdy, Mathematical analysis and simulations for a Caputo-Fabrizio fractional COVID-19 model, Partial Differential Equations in Applied Mathematics, 2023, 8, 1–12.

v. R. P. Agarwal, A. M. A. El-Sayed and S. M. Salman, Fractional-order Chua system discretization, bifurcation and chaos, Advances in Difference Equations, 2013, 2013, 1–13. doi: 10.1186/1687-1847-2013-1

vi. E. Ahmed, A. Hashish and F. A. Rihan, On fractional order cancer model, Journal of Fractional Calculus and Applied Analysis, 2012, 3(2), 1–6. A. M. Arafa, S. Z. Rida and M. Khalil, Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection, Non-linear Biomedical Physics, 2012, 6(1), 1–7. doi: 10.1186/1753-4631-6-1

vii. N. Bellomo, A. Bellouquid, J. Nieto and J. Soler, Multiscale biological tissue models and flux-limited chemotaxis for multicellular growing systems, Mathematical Models and Methods in Applied Sciences, 2010, 20(7), 1179–1207. doi: 10.1142/S0218202510004568

viii. M. Cosnard and P. Fraigniaud, Finding the roots of a polynomial on an MIMD multicomputer, Parallel Computing, 1990, 15(1–3), 75–85. doi: 10.1016/0167-8191(90)90032-5

ix. K. Diethelm, J. Ford and A. Freed, Multi-order fractional differential equations and their numerical solution, Applied Mathematics and Computation, 2004, 154, 621–640. doi: 10.1016/S0096-3003(03)00739-2 El-Mesady, A. Elsonbaty and W. Adel, On nonlinear dynamics of a fractional order monkeypox virus model, Chaos Solitons and Fractals, 2022, 164, 112716. doi: 10.1016/j.chaos.2022.112716

x. El-Mesady, A. A. Elsadany, A. M. S. Mahdy and A. Elsonbaty, Nonlinear dynamics and optimal control strategies of a novel fractional-order Lumpy Skin disease model, Journal of Computational Science, 2024, 102286.

xi. A. Elsadany and A. E. Matouk, Dynamical behaviors of fractional-order Lotka-Volterra predator-prey model and its discretization, Applied Mathematics and Computation, 2015, 49, 269–283. doi: 10.3969/j.issn.1006-6330.2015.03.002

xii. M. A. El-Sayed and S. M. Salman, On a discretization process of fractional order Riccati differential equation, Journal of Fractional Calculus and Application, 2013, 4, 251–259.

xiii. M. El-Shahed, J. J. Nieto, A. M. Ahmed and I. M. E. Abdelstar, Fractional-order model for biocontrol of the lesser date moth in palm trees and its discretization, Advances in Difference Equations, 2013, 2013, 1–16. doi: 10.1186/1687-1847-2013-1 El-Sonbaty, M. Alharbi, A. El-Mesady and W. Adel, Dynamical analysis of a novel discrete fractional lumpy skin disease model, Partial Differential Equations in Applied Mathematics, 2024, 9, 1–12.

xiv. M. I. A. Fathoni, Gunardi, F. A. Kusumo and S. H. Hutajulu, Mathematical model analysis of breast cancer stages with side effects on heart in chemotherapy patients, in AIP Conference Proceedings, AIP Publishing LLC, 2019, 2192(1), 060007.

xv. R. L. Fournier, Basic transport phenomena in biomedical engineering, New York: Taylor and Francis, 2007, 1–611. Gokdogan, A. Yildirim and M. Merdan, Solving a fractional order model of HIV infection of CD+ Tcells, Mathematical and Computer Modelling, 2011, 54(9–10), 2132–2138. doi: 10.1016/j.mcm.2011.05.022

xvi. Higazy, A. El-Mesady, A. M. S. Mahdy, S. Ullah and A. Al-Ghamdi, Numerical approximate solutions and optimal control on the deathly lassa hemorrhagic fever disease in pregnant women, Journal of Function Spaces, 2021, 2021, 1–15.

xvii. D. Kirschner and J. C. Panetta, Modeling immunotherapy of the tumor-immune interaction, Journal of Mathematical Biology, 1998, 37(3), 235–252. doi: 10.1007/s002850050127

xviii. K. Kretzschmar, Cancer research using organoid technology, Journal of Molecular Medicine, 2021, 99, 501–515. doi: 10.1007/s00109-020-01990-z

xix. Amr M.S. Mahdy, Norah Almalki, Mahmoud Higazy,

xx. A general fractional breast cancer model: Model graph energy, Caputo-Fabrizio derivative existence and uniqueness plus numerical simulation,Partial Differential Equations in Applied Mathematics,Volume 10,2024,100723,ISSN 2666-8181,https://doi.org/10.1016/j.padiff.2024.100723.

xxi. V. Mercurio, F. Pirozzi, E. Lazzarini, G. Marone, P. Rizzo, G. Agnetti, et al.Models of heart failure based on the cardiotoxicity of anticancer drugs J Card Fail, 22 (6) (2016), pp. 449-458

xxii. M. Caputo, M. Fabrizio A new definition of fractional derivative without singular kernel Progr. Fract. Differ. Appl, 1 (2) (2015), pp. 1-13

xxiii. Zhuang and Cressie, 2014 L. Zhuang, N. Cressie Bayesian hierarchical statistical SIRS models Stat. Methods Appt., 23 (2014), pp. 601-646

xxiv. Yadav et al., 2022 S.K. Yadav, S. Zhao, Y. Akhter Preliminary estimation of the basic reproduction number of novel coronavirus (2019-nCoV) in China, from 2019 to 2020: a data-driven analysis in the early phase of the outbreak Intern. J. Infectious Diseases, 115 (2022), pp. 70-71, 10.1016/j.ijid.2021.12.310