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Solvability and Nilpotency of Finite Groups

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ABSTRACT :

If G be a group and H its any subgroup. S is a right transversal to a subgroup H of group G. Then S forms a right quasigroup with identity with respect to the binary operation $\circ$ given by $x \circ y = S \cap Hxy$. We find some relationship between solvability and Nilpotency for finite groups.

Keywords *- Finite Groups, Nilpotency, Solvability*

INTRODUCTION

From the beginning of the last century in mathematics the study of groups whose each proper subgroups have important properties. A Dedekind group is a group whose each subgroup are normal. Groups all of whose proper subgroups are abelian was studied by Miller and Moreno [1]. They proved that such finite groups are solvable. Such an infinite group can even be simple. Baer [2] showed that a non-abelian group whose all subgroups are embedded as normal subgroups is a direct product of quaternion group, an elementary abelian 2-group and an abelian group all of whose elements are of odd order. A subgroup H of a group G is said to be stable if for any pair S_1 and S_2 of right transversals to H in G, the group torsions $G_{S_1} = (\langle S_1\rangle \cap H)/(\langle S_1\rangle \cap Core_G(H))$ and $G_{S_2} = (\langle S_2\rangle \cap H)/(\langle S_2\rangle \cap Core_G(H))$ are isomorphic. Clearly, any normal subgroup of G is stable. A finite solvable group all of whose proper subgroups are stable is a Dedekind group [3, Theorem 2.4].

PRELIMINARIES:

Solvable Groups

Let G be a group. Define subgroups G^n of G inductively as follows: Define $G^1 = G'$ where G' is the derived subgroup of the group G, being generated by all commutators in G thus $G' = [G,G]$. Assuming that G^n has already been defined, define $G^{m+1} = [G^m, G^m]$.

Thus we get a series $G = G^0 \trianglerighteq G^1 \trianglerighteq G^2 \trianglerighteq \cdots \trianglerighteq G^m \trianglerighteq G^{m+1} \cdots$

This series is called the derived series or the commutator series of G, G^n is called nth term of the derived series, although it need not reach 1 or even terminate. Of course all the factors $G^{(n)}/G^{(n+1)}$ are abelian groups: the first of these, G/G', is of particular importance and if often written G_{ab} since it is the largest abelian quotient group of G.

THE LOWER AND UPPER CENTRAL SERIES

Let G be a group. There is another natural way of generating a descending sequence of commutators subgroups of a group, by repeatedly commuting with G. Define subgroups $L_n(G)$ inductively as follows:

Define $L_0(G) = G, L_1(G) = [G, L_0(G)] = [G, G] = G'$. Supposing that $L_n(G)$ has already been that defined, define $L_{n+1}(G) = [G, L_n(G)]$. Clearly each $L_n(G)$ is normal in G. Thus we get a descending series $G = L_0(G) \trianglerighteq L_1(G) \trianglerighteq L_2(G) \trianglerighteq \cdots \trianglerighteq L_n(G) \trianglerighteq \cdots$ of G is called lower central series of G. Notice that $L_n(G)/L_{n+1}(G)$ lies in the center of $G/L_{n+1}(G)$ and that each $L_n(G)$ is fully invariant in G. Like the derived series the lower central series does not reach $\{e\}$ in general and $L_1(G)$ is the first term of lower central series where as $G^{(0)}$ is the first term of the derived series.

Define normal subgroups $Z_n(G)$ of G inductively as follows: Define $Z_0(G) = \{e\}, Z_1(G) = Z(G)$ the center of G. Observe that $Z_1(G)/Z_0(G) = Z(G/Z_0(G))$ the center of $G/Z_0(G)$. Supposing that $Z_n(G)$ has already defined, define $Z_{n+1}(G)$ by equation $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$

Thus we get an ascending series $\{e\} = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \cdots \trianglelefteq Z_n(G) \trianglelefteq \cdots$ of normal subgroups of G. This is called the upper central series of G. It is dual to the lower central series in the same sense that the center is dual to the commutator subgroup. Each $Z_n(G)$ is characteristic but not necessarily fully-invariant in G. This series need not reach G, but if G is finite, the series terminate to a subgroup called hyper center.

NILPOTENT GROUPS

A group G is said to be nilpotent if $L_n(G) = \{e\}$ or equivalently $Z_n(G) = G$ for some n. A group G is said to be nilpotent of class n if $L_n(G) = \{e\}$ but $L_{n-1}(G) \neq \{e\}$ or equivalently $Z_{n-1}(G) \neq G$.

A group G is called nilpotent if it has a central series that is, a normal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that G_{i+1}/G_i is contained in the center of $G/G_i \ \forall i$

RESULT

Theorem 2.1 : If G be a finite solvable group. Then the following statements are equivalent.

1. Given a maximal subgroup M of G and $G = MK$ for some subgroup $K, G = Core_G(M)K$.
2. All maximal subgroups of G are strongly stable.
3. All maximal subgroups of G are stable.
4. G is nilpotent.
5. All Sylow subgroups of G are stable.
6. All Sylow subgroups are strongly stable.
7. Given a Sylow subgroup P and $G = PK$ for some subgroup $K, G = Core_G(P)K$.
8. All maximal subgroups of G are pre-normal.
9. All maximal subgroups of G are strongly pre-normal.
10. Given a maximal subgroup M of G and $G = MK$ for some subgroup K of $G, K \cap M \trianglelefteq K$.

11. All Sylow subgroups of G are pre-normal.
12. All Sylow subgroups of G are strongly pre-normal.
13. Given a Sylow subgroup P of G and $G = PK$ for some subgroup K of $G, K \cap P \trianglelefteq K$.

Proof: $(4 \Leftrightarrow 3)$ A finite nilpotent subgroup has all its maximal subgroups normal and so stable. Conversely, suppose that all maximal subgroups of a finite solvable group G are stable. Then we have to show that G is nilpotent. Suppose contrary. Let G be a minimal counter example. Then there exists a maximal subgroup M of G which is not normal. Suppose that $Core_G(M) \neq \{e\}$. Then $G/Core_G(M)$ is also a solvable group of smaller order all of whose maximal subgroups are stable. By the fact that G is minimal counter example, $G/Core_G(M)$ is nilpotent and so $M/Core_G(M)$ being maximal subgroup of $G/Core_G(M)$ is normal. But, then M is normal, a contradiction to the supposition. Thus $Core_G(M) = \{e\}$. Let G be a nontrivial minimal normal subgroup of G. Then $G = ML$ and $L \cap M = \{e\}$. Since M is stable and L can be taken as a right transversal which is a group, group torsion of all right transversals of M in G will be trivial. But, then M will be normal, a self-contradiction.
In order to prove rest statements we need further result:

Theorem 2.2 Let H be a subgroup of G. Then H is a stable subgroup of G if and only if

$$(\langle S_1\rangle \cap H)/(\langle S_1\rangle \cap Core_G(H)) \cong (\langle S_2\rangle \cap H)/(\langle S_2\rangle \cap Core_G(H))$$

for any pair S_1 and S_2 of right transversals of H in G.

Theorem 2.3 Let H be a subgroup of a finite group G. Then the following conditions are equivalent.[3]

(i.) H is strongly stable subgroup.
(ii.) $G = HK \Rightarrow H = Core_G(H)(K \cap H)$.
(iii.) $G = HK \Rightarrow G = Core_G(H)K$.

Theorem 2.4 Suppose that G is a finitely generated soluble group. If G is not nilpotent, then it has a finite image that is not nilpotent.

PROOF OF THEOREM 2.1

$(1 \Rightarrow 2)$ Result follows from theorem 2.2
$(3 \Rightarrow 4)$ It G be a finitely generated solvable group all of whose maximal subgroups are stable and K be a its finite quotient. Then K is finite solvable and all its maximal subgroups will be stable. But, then K will be nilpotent. The result follows from theorem which states that a finitely generated non-nilpotent group has a finite image that is not nilpotent. The rest of equivalences follows as in the theorem 2.3
$(2 \Rightarrow 3)$ A subgroup H of G will be called a strongly stable subgroup if whenever $G = K_1H = K_2H$, where K_1 and K_2 are subgroups of G, we have $(K_1 \cap H)/(K_1 \cap Core_G(H)) \cong (K_2 \cap H)/(K_2 \cap Core_G(H))$ or equivalently,
$G = KH \Rightarrow (K \cap H)/(K \cap Core_G(H)) \cong H/Core_G(H)$
And every nilpotent group is solvable.

$(4 \Rightarrow 1)$If M is a maximal subgroup of a finite nilpotent group, then $Core_G(M) = M$. Thus conditions from 1 to 4 are equivalent.
$(4 \Rightarrow 5)$ In a finite nilpotent group all Sylow subgroups are normal and so stable.
$(5 \Rightarrow 1)$Since G is solvable, given a Sylow p-subgroup P, it has a complement subgroup Q such that $G = PQ, P \cap Q = \{e\}$. Thus Q can be taken as a right transversal whose group torsion is trivial. But, then since P is stable, group torsion of all right transversals of P in G are trivial. This means that P is normal.

The proof of equivalence of 4, 5, 6 and 7 is as in the proof of equivalence

of 4, 3, 2 and 1.
$(4 \Rightarrow 8)$ Obvious from definition.
$(8 \Rightarrow 1)$Assume that all maximal subgroups of G are pre-normal. To prove that G is nilpotent. Suppose not and let G be a minimal counter example. Then there exists a maximal subgroup M of G which is not normal. If $Core_G(M) \neq \{e\}$, then $G/Core_G(M)$ is also a solvable group of smaller order all of whose maximal subgroups are pre-normal. Since G is minimal counter example, $G/Core_G(M)$ is nilpotent. Hence all maximal subgroup of $G/Core_G(M)$ are normal. In particular, M is normal, a contradiction. Thus $Core_G(M) = \{e\}$.As in the proof of $1 \Rightarrow 2$, take a nontrivial minimal normal subgroup L of G. Then $G = ML$ and $L \cap M = \{e\}$. Then L can be taken as a right transversal of M in G. Clearly, $M(L) \cong L$. Thus, since M is pre normal, for any right transversal S of M in G, $M(S) \cong L$. In particular $|M(S)| = |L| = |S|$. This shows that S is a group and so M is normal, a contradiction.

Theorem 2.5 Let G be a nilpotent group and H a subgroup of G then for every transversal S to H in $G \exists x \in S - \{e\}$ such that $x\theta h = x \forall h \in H$. In particular $x\theta f(y,z) = x \forall y,z \in S$ and $G_S \subseteq Sym(S - \{e,x\})$ and $G_S \subset N_{G_S S}(G_S)$.

Proof: Let G be a nilpotent group, take H be any subgroup of G then $H \subset N_G(H) \exists g \in N_G(H) - H$ such that $gHg^{-1} = H$ i.e.$\exists g \in G - H$ such that $ghg^{-1} \in H$ i.e. $ghg^{-1} = h' \forall h \in H$ and $h' \in H$.
Since $g \in G = HS \Rightarrow g = h_1 x$ where $h_1 \in H, x \in S - \{e\}$.
Since $g \notin H$ so that $(h_1 x)h(h_1 x)^{-1} = h'$
$h_1 xhx^{-1}h_1^{-1} = h'$ or $xhx^{-1} = h''$ for some $h'' \in H$.
$\Rightarrow xh = h''x$
$\Rightarrow x\theta h = x \forall h \in H$
$i.e., \exists x - S - \{e\}$ such that $x\theta h = x$.
In particular $x\theta f(y,z) = x \forall y,z \in S.$.
Since $f(S \times S) \subseteq H$. If q be the permutation representation of $f(S \times S)$ on S defined by $q(f(y,z)) = f^S(y,z)$ then $\exists$ a homomorphism $\psi : f(S \times S)S \to G_S S$ defined by $\psi(hx) = q(h)x$. Then we see that
$x\theta^S f^S(y,z) = x \forall y,z \in S$

i.e. $f^S(y,z)(x) = x \forall y,z \in S$

so that $f^S(y,z) \in Sym(S-\{e,x\}) \forall y,z \in S$

$\Rightarrow G_S \subseteq Sym(S-\{e,x\})$

Now

$xf^S(y,z)x^{-1}$

$= \sigma_x^S(f^S(y,z))x\theta^S f^S(y,z)x^{-1}$

$= \sigma_x^S(f^S(y,z))x(f^S(x',x))^{-1}x'$

$= \sigma_x^S(f^S(y,z))\sigma_x^S(f^S(x',x))^{-1} f^S\left(x\theta^S(f^S(x',x^{-1}))^{-1}, x'\right)x\theta^S(f^S(x',x))^{-1} \circ x'$

$= \sigma_x^S(f^S(y,z))\sigma_x^S(f^S(x',x))^{-1} f^S(x,x')x \circ x'$

$= \sigma_x^S(f^S(y,z))\sigma_x^S(f^S(x',x))^{-1} f^S(x',x) \in G_S$

So $G_S \subset N_{G_S S}(G_S)$.

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